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FONCTIONS ELLIPTIQUES, FRACTIONS CONTINUES ET PERMUTATIONS

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Abstract.

The Taylor coefficients of the Jacobian elliptic functions are shown to count classes of permutations with a simple repetitive order pattern. The proof relies on the use of enumerative properties of continued fractions, and on a mapping between path diagrams and permutations.

Résumé.

Nous montrons que les coefficients de Taylor des fonctions elliptiques de Jacobi comptent diverses classes de permutations ayant des propriétés ordinales simples. La preuve de ces résultats repose sur l'utilisation de la combinatoire des fractions continues, des chemins et des permutations.

1. INTRODUCTION:

This paper is relative to some *enumerative properties* of the *Jacobian elliptic functions* sn, cn, dn. For a fixed "modulus" α, the function sn is defined as the inverse of an elliptic integral:

$$sn(z,\alpha)=y \quad \text{iff} \quad z=\int_0^y (1-t^2)^{-\frac{1}{2}}(1-\alpha^2t^2)^{-\frac{1}{2}}\,dt\ . \tag{1a}$$

The other functions cn, dn are given by

$$cn(z,\alpha)=(1-sn^2(z,\alpha))^{\frac{1}{2}} \tag{1b}$$

$$dn(z,\alpha)=(1-\alpha^2 sn^2(z,\alpha))^{\frac{1}{2}}\ . \tag{1c}$$

Corresponding Taylor expansions have long been known [5,II,p. 344], and one has:

$$sn(z,\alpha)=z-(1+\alpha^2)\frac{z^3}{3!}+(1+14\alpha^2+\alpha^4)\frac{z^5}{5!}+\cdots \tag{2a}$$

$$cn(z,\alpha)=1-\frac{z^2}{2!}+(1+4\alpha^2)\frac{z^4}{4!}-(1+44\alpha^2+16\alpha^4)\frac{z^6}{6!}+\cdots \tag{2b}$$

$$dn(z,\alpha)=1-\alpha^2\frac{z^2}{2!}+\alpha^2(4+\alpha^2)\frac{z^4}{4!}-\alpha^2(16+44\alpha^2+\alpha^4)\frac{z^6}{6!}+\cdots \tag{2c}$$

The question of the possible combinatorial significance of the integer coefficients appearing in the Taylor expansions of the Jacobian elliptic functions has first been raised by Schutzenberger. Indeed, these integers generalise the Euler numbers, *i.e.* the coefficients of $\tan(z)$, $\sec(z)$ whose relation to permutations has been known since André [1]. The first combinatorial interpretation has been given by Viennot[17], and is expressed in terms of so-called Jacobi permutations. Flajolet [6] has shown the coefficients of cn to count classes of alternating (up-and-down) permutations based on the parity of peaks. Last Dumont [4] discovered some further relations between these functions and the cycle structure of permutations.

In this paper, we give a simple interpretation of the elliptic functions as generating functions of *doubled permutations*. Such permutations are essentially defined by the property that, for all k, elements $2k-1$ and $2k$ are of the same ordinal type.

Elements in a permutation can be classified according to their *ordinal type* into four categories:

Definition: *Let* $s=s_1s_2s_3\cdots s_n$ *be a permutation of* $[1..n]$. *An element* s_i *is:*

a peak iff $s_{i-1}<s_i>s_{i+1}$;

a valley iff $s_{i-1}>s_i<s_{i+1}$;

a double rise iff $s_{i-1}<s_i<s_{i+1}$;

a double fall iff $s_{i-1}>s_i>s_{i+1}$;

with the convention that $s_0=s_{n+1}=0$. *An element which is either a valley or a double rise is called a rise; an element which is either a peak or a double fall is called a fall.*

Thus an element s_i $(1\le i\le n)$ is a rise if $s_{i-1}<s_i$; it is a fall otherwise.

Definition: *A permutation is said to be a doubled permutation iff for all* i, *elements (i.e.* values) $2i+1$ *and* $2i+2$ *are of the same ordinal type.*

For instance, 6 7 1 5 12 9 3 8 4 11 2 13 10 is a *doubled permutation*, since pairs 1 2, 3 4, $\cdots$, 11 12 share the same ordinal type (in this case of an odd number of elements, the last element _ here 13_ is necessarily a peak).

We propose to prove here:

Theorem: *The coefficient of* $(-1)^n \frac{z^{2n+1}}{(2n+1)!}\alpha^{2k}$ *in* $sn(z,\alpha)$ *counts the number of (odd) doubled permutations over* $2n+1$ *having* $2k+1$ *falls.*

The coefficient of $(-1)^n \frac{z^{2n}}{(2n)!}\alpha^{2k}$ *in* $cn(z,\alpha)$ *counts the number of (odd) doubled permutations over* $2n+1$ *terminated by* $2n+1$ *and having* $2k+1$ *falls.*

Notice for the interpretation of the coefficients of cn that one could also take the alternative convention that for even n, $s_{n+1}=\infty$. The coefficients of cn would in that case count the number of (even) doubled permutations with a prescribed number of falls.

As a check to this theorem we see that for $n=2$, the doubled permutations classified according to their numbers of falls are:

1 fall: 12345;

3 falls: 31425, 32415, 41325, 42315, 31524, 32514, 41523, 42513, 51324, 52314, 51423, 52413, 12543, 34521;

5 falls: 54321.

This list is in accordance with the fact that the coefficient of $\frac{z^5}{5!}$ in $sn(z,\alpha)$ is $1+14\alpha^2+\alpha^4$. Similarly retaining only those permutations in the above list that end with a 5, we find 1 with 1 fall, and 4 with 3 falls, in accordance with the value of the Taylor coefficient of $\frac{z^4}{4!}$ in $cn(z,\alpha)$ which is $1+4\alpha^2$.

Our proof relies partly on the general enumerative properties of continued fractions presented in [6] in connection with path diagrams. Partly, it depends on a correspondence between path diagrams and permutations discovered by Françon and Viennot [7].

2. ELLIPTIC FUNCTIONS AND CONTINUED FRACTIONS:

It has been found by Stieltjes and Rogers [14, 15] that the Laplace-Borel transforms of sn and cn defined by:

$$S_1(z,\alpha) = \int_0^\infty e^{-t}\, sn(zt,\alpha)\, dt$$

$$C_0(z,\alpha) = \int_0^\infty e^{-t}\, cn(zt,\alpha)\, dt \;,$$

i.e. the series obtained from (2) by replacing $\frac{z^n}{n!}$ by z^n, have continued fractions of a simple form. Rogers' proof which is the simpler one being partly faulty, we shall briefly summarise the steps of a correct derivation here.

Theorem 1: *The functions $S_1(z,\alpha)$ and $C_0(z,\alpha)$ have the following continued fraction expansions:*

$$S_1(z,\alpha)=\cfrac{z}{1+z^2(1+\alpha^2)-\cfrac{1.2^2.3.\alpha^2z^4}{1+3^2z^2(1+\alpha^2)-\cfrac{3.4^2.5.\alpha^2z^4}{1+5^2z^2(1+\alpha^2)-\cdots}}}$$

$$C_1(z,\alpha)=\cfrac{1}{1+z^2-\cfrac{1^2.2^2.\alpha^2z^4}{1+z^2(3^2+2^2\alpha^2)-\cfrac{3^2.4^2.\alpha^2z^4}{1+z^2(5^2+4^2\alpha^2)-\cdots}}}$$

Proof: (i) For any integer k, we define the function:

$$S_k(z,\alpha)=\int_0^\infty e^{-t}sn^k(zt,\alpha)\,dt\ .$$

Through integration by parts, appealing to the differential equation satisfied by sn, one gets:

$$S_1(z,\alpha)=z-z^2(1+\alpha^2)S_1(z,\alpha)+2\alpha^2z^2S_3(z,\alpha)$$

and, for $n>1$,

$$S_n(z,\alpha)=z^2n(n-1)S_{n-2}(z,\alpha)-n^2z^2(1+\alpha^2)S_n(z,\alpha)+n(n+1)\alpha^2z^2S_{n+2}(z,\alpha)\ .$$

This can be transformed into the equivalent forms:

$$\frac{S_n}{S_{n-2}}=\frac{n(n-1)z^2}{1+n^2z^2(1+\alpha^2)-n(n+1)\alpha^2z^2\frac{S_{n+2}}{S_n}}\ .$$

whence by iteration of these functional relations the expansions

$$S_1=\cfrac{z}{1+z^2(1+\alpha^2)-\cfrac{1.2^2.3.\alpha^2z^4}{1+3^2z^2(1+\alpha^2)-\cfrac{3.4^2.5.\alpha^2z^4}{1+5^2z^2(1+\alpha^2)-\cdots}}}$$

(ii) Similarly defining

$$C_k(z,\alpha)=\int_0^\infty e^{-t}cn(zt,\alpha)sn^k(zt,\alpha)\,dt\ ,$$

one finds:

$$C_0=1-z^2C_0+2\alpha^2z^2C_2$$

and for $n>1$,

$$C_n=n(n-1)z^2C_{n-2}-((n+1)^2+n^2\alpha^2)z^2C_n+(n+1)(n+2)\alpha^2z^2C_{n+2}\ .$$

whence, by the same method as before,

$$C_0=\cfrac{1}{1+z^2-\cfrac{1^2.2^2.\alpha^2z^4}{1+(3^2+2^2\alpha^2)z^2-\cdots}}\ . \qquad \square$$

3. ELLIPTIC FUNCTIONS AND PATH DIAGRAMS

To interpret the continued fraction expansions combinatorially, we first need the concept of a *path diagram.*

Definition: *A path of length n is a map*

$$p : \{0,1,2,\cdots,n\} \to \mathbf{N}, \quad n \in \mathbf{N},$$

such that, for all i, $0 \le i < n$, $|p(i+1)-p(i)| \le 1$, *and* $p(0)=p(n)=0$. *A step of a path is a pair* $(p(i),p(i+1))$; *such a step is said to start at level* $p(i)$ *and to end at level* $p(i+1)$.

A step is called ascending ("a"), descending ("d") or level ("l") iff respectively:

$$p(i+1)=p(i)+1, \quad p(i+1)=p(i)-1, \quad p(i+1)=p(i)\,.$$

For convenience in our subsequent treatment, we consider *labelled paths* in which a level step may be labelled in two different ways, either with a *plus* ("l^+") or a *minus* ("l^-").

When graphed in the cartesian planes, path are thus sequences of steps that are positive, start at level 0, and end at level 0. Accordingly, a path can be conceived of as a sequence of steps, *i.e.* a word over the alphabet $\{a,d,l\}$ or, when labelled, over the alphabet $\{a,d,l^+,l^-\}$. For instance, the mapping p defined by

$$p(0)=0, p(1)=1, p(2)=1, p(3)=2, p(4)=1, p(5)=0, p(6)=0,$$

is a path which may also be written as the sequence of steps $a\ l\ a\ d\ d\ l$. A possible labelling of level steps is $a\ l^+ a\ d\ d\ l^-$.

Definition: *Any application*

$$P\ ;\ \{a,d,l^+,l^-\} \times \mathbf{N} \to \mathbf{N}$$

is called a possibility rule. A path diagram corresponding to possibility rule P *is defined as a couple consisting of a first component that is a path and a second component that is a sequence of integers (the valuation of the path diagram) obeying dominance rules specified by* P*: if the* j*-th step in the path is an* ω *step starting at level* k*, then the* j*-th component in the valuation is restricted to be in the integer interval* $[1..P(\omega,k)]$.

Path diagrams can also be encoded as words over an alphabet:

$$X = \{\, \omega_{i,j} \mid \omega \in \{a,d,l^+,l^-\}\,;\, i,j \in \mathbf{N} \,\}\,.$$

The following theorem was proved in [6]:

Theorem 2: *The characteristic series of path diagrams with possibility set* P *is given by the continued fraction:*

$$\cfrac{1}{1-\sum l^+_{0,j}-\sum l^-_{0,j}-\cfrac{\sum a_{0,j}\sum d_{1,j}}{1-\sum l^+_{1,j}-\sum l^-_{1,j}-\cfrac{\sum a_{1,j}\sum d_{2,j}}{1-\sum l^+_{2,j}-\sum l^-_{2,j}-\cfrac{\sum a_{2,j}\sum d_{3,j}}{\cdots}}}}$$

where in each sum $\sum \omega_{i,j}$, *the index* j *is restricted to the interval* $[1..P(\omega,i)]$.

(See also [8, 9, 12, 16] for related results). This theorem provides for a variety of applications based on the possibility of "marking" various parameters of path diagrams. As an immediate application here, we get:

Proposition 1: *The coefficient of* $z(-z^2)^n\alpha^k$ *in* $S_1(z,\alpha)$ *counts the number of path diagrams of length* n, *formed with a total of* k *steps of type either* a *or* l^-, *corresponding to the possibility rule:*

$$\pi(a,k)=(2k+1)(2k+2)\,;\ \pi(d,k)=2k(2k+1)\,;\ \pi(l^+,k)=\pi(l^-,k)=(2k+1)^2\,.$$

The coefficient of $(-z^2)^n\alpha^{2k}$ *in* $C_0(z,\alpha)$ *counts the number of path diagrams of length* n, *formed with a total of* k *steps of type either* a *or* l^-, *corresponding to the possibility rule:*

$$\pi'(a,k)=(2k+1)(2k+2)\,;\ \pi'(d,k)=2k(2k-1)^2\,;\ \pi'(l^-,k)=(2k)^2\,;\ \pi'(l^+,k)=(2k+1)^2\,.$$

Proof: In the case of *e.g.* sn, consider the morphism μ defined by

$$\mu(a_{i,j})=-z^2\alpha^2\,;\ \mu(d_{i,j})=-z^2\,;\ \mu(l^+_{i,j})=-z^2\,;\ \mu(l^-_{i,j})=-z^2\alpha^2\,.$$

It transforms the characteristic function of path diagrams into the generating function, with $(-z)^2$ marking length and α^2 marking steps of type ascent a or level l^-. The continued fraction becomes the expansion of $S_1(z,\alpha)/z$. $\square$

4. ELLIPTIC FUNCTIONS AND PERMUTATIONS:

Path diagrams of the type previously encountered do not belong to some of the classes known to be in correspondence with simple combinatorial structures [6, 7] like permutations, set partitions... . We propose to reduce them to one of the cases covered by the following theorem of [7].

Theorem 3: *(i) Permutations over* $n+1$ *are in bijective correspondence with path diagrams of length* n *with the possibility rule:*

$$\Pi(a,k)=k+1\,;\ \Pi(d,k)=k+1\,;\ \Pi(l^+,k)=\Pi(l^-,k)=k+1\,;$$

(ii) permutations over n *(equivalently permutations over* $n+1$ *ending with* $n+1$*) are in correspondence with path diagrams of length* n *obeying the possibility rule :*

$$\Pi'(a,k)=k+1\,;\ \Pi'(d,k)=k\,;\ \Pi'(l^+,k)=k+1\,;\ \Pi'(l^-,k)=k.$$

Furthermore the valleys (respectively peaks, double rises and double falls) of the permutations correspond to the ascending steps (respectively descents, level steps l^+ *and level steps* l^-*) of the associated path diagram.*

We now introduce *doubled paths* which have the property of allowing us to reduce path diagrams of Proposition 1 to a particular subset of the Françon-Viennot paths of Theorem 3:

Definition: *A path is a doubled path iff it belongs to the set*

$$\{aa,dd,l^+l^+,l^-l^-\}^*\,.$$

In other words, in a doubled path, a step of odd position is always followed by a step of the same type. We can now state:

Proposition 2: *The coefficient of* $z(-z^2)^n\alpha^{2k}$ *in* $S_1(z,\alpha)$ *counts the number of path diagrams of length* $2n$ *corresponding to the possibility set* Π, *having a total of* $2k$ *steps of type* a *or* l^+, *whose associated path is a doubled path.*

The coefficient of $(-z^2)^n\alpha^{2k}$ *in* $C_0(z,\alpha)$ *counts the number of path diagrams of length* $2n$ *corresponding to the possibility set* Π', *having a total of* $2k$ *steps of type either* a *or* l^+, *whose associated path is a double path.*

Proof: Grouping steps 2 by 2 in a path diagram with possibility set Π (resp. Π') and a path which is a double path yields an unconstrained path diagram with possibility set π (resp. π'). The result then follows by Proposition 1. $\square$

We are now in a position to conclude the proof of our main theorem. By Theorem 3 (and the following remarks), doubled path diagrams are in correspondence with doubled permutations and are counted by the coefficients of the elliptic functions as Proposition 2 shows.

A somewhat more complicated interpretation of the coefficients of $sn^k(z,\alpha)$ and $cn(z,\alpha)sn^k(z,\alpha)$ could also be given along similar lines.

5. CONCLUSION

Although the combinatorial properties of the elliptic functions are not fully understood, all known interpretations including the one given here are based on the introduction of some sort of parity constraint on permutations. The elliptic functions are also expressible in terms of theta functions, whose relation to integer partitions is well known [3]; this seems to indicate the possible existence of relations between permutations and partitions, which are yet to be discovered.

As a companion to our interpretation, we can mention the dual result of Carlitz, Scoville and Vaughan that the coefficient of $\frac{z^n}{(n!)^2}$ in

$$\left[\sum(-1)^n \frac{z^n}{(n!)^2}\right]^{-1}$$

is equal to the number of pairs of permutations in which rises at corresponding places are forbidden. One finds in Carlitz's survey paper [2] a wealth of enumerations dealing with ordinal properties of permutations.

The reader could consult the recent conference proceedings [13] (and references therein) for similar approaches to combinatorial enumerations. A number of related combinatorial results on continued fractions are presented in the book of Jackson and Goulden [9], as well as in [10, 11, 18].

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